The Real Transform: Computing Positive Solutions of Fuzzy Polynomial Systems

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Abstract: This paper presents an efficient method for finding the positive solutions of polynomial systems whose coefficients are symmetrical *L-R* fuzzy numbers with bounded support and the same bijective spread functions. The positive solutions of a given fuzzy system are deduced from the ones of another polynomial system with real coefficients, called the real transform. This method is based on new results that are universal because they are independent from the spread functions. We propose the real transform $\mathcal{T}(E)$ of a fuzzy equation (E), which positive solutions are the same as those of (E). Then we compare our approach with the existing method of the crisp form system.

1 INTRODUCTION

Modeling problems with uncertain data has important applications in engineering, economics and social sciences (Aluja et al., 1994; Lodwick, 2007). As fuzzy functions involved in equations can be approximated by fuzzy polynomials (Liu, 2002; Abbasbandy and Amirfakhrian, 2006), the problem of solving fuzzy polynomial systems is of main importance and has motivated many studies, among them (Buckley and Qu, 1990; Buckley and Eslami, 1997; Buckley et al., 2002; Kajani et al., 2005).

Some problems are modeled by a system of continuous fuzzy valued functions defined over $\mathbb{R}^n$ (see (Lodwick and Santos, 2003)); Liu (Liu, 2002) proposes a polynomial interpolation that provides an interface between solving these problems and solving fuzzy polynomial systems. Therefore several authors have been interested in searching for real solutions of polynomial equations with fuzzy coefficients. Note that their work, and ours too, is based only on Zadeh's extension principle. The methods of resolution were initially based on local techniques, firstly for one univariate polynomial (Abbasbandy and Otadi, 2006; Amirfakhrian, 2008; Rouhparvar, 2007) and later for multivariate systems (Abbasbandy et al., 2008; Ahmad et al., 2011).

Abbasbandy et al. (Abbasbandy et al., 2008) study in particular systems of the form

$$\begin{cases} \widetilde{a}_{1,1}xy+\widetilde{a}_{1,2}x^2y^2+\cdots+\widetilde{a}_{1,d}x^dy^d & = & \widetilde{a}_{1,0} \\ \widetilde{a}_{2,1}xy+\widetilde{a}_{2,2}x^2y^2+\cdots+\widetilde{a}_{2,d}x^dy^d & = & \widetilde{a}_{2,0} \end{cases}$$

and they present several systems coming from applications like cross location of quadratic surfaces or in economics.

Recently, a global approach using classical algebraic techniques has been developed (Molai et al., 2013; Farahani et al., 2015; Boroujeni et al., 2016; Farahani et al., 2019). Indeed, despite a name that may be confusing, fuzzy numbers benefit from a perfectly formal definition. We revisit this approach and we significantly strengthen it. In the past, both local and global approaches focused on so-called *triangular* fuzzy numbers, that is, with linear spread functions. The results presented here consider more generally fuzzy coefficients with bounded support and the same bijective spread functions such as, for example, quadratic fuzzy numbers, that have quadratic spread functions. As previous works on this topic, our notion of solution lies on the equality of membership functions.

In a fuzzy algebraic system, the fuzzy coefficients (coming from the experiments) are generally given under a representation called "tuple". Although the tuple representation is formal, it cannot be handle by usual algebraic methods (Gröbner bases (Becker and Weispfenning, 1993), triangular decomposition (Aubry and Maza, 1999), rational univariate representation (Rouillier, 1999), ...) to solve the system.

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Nevertheless, for any fuzzy number with bounded support and of bijective spread functions, this tuple representation is transformable into another representation called "parametric", where the coefficients are no longer fuzzy but real. We give the expression of the parametric representation as a function of the inverse of its spread functions (see Proposition 1).

We show how finding the positive solutions of a system (S) of s equations with k variables and with symmetrical fuzzy coefficients reduces to computing positive solutions of a system formed by $3s$ equations of k variables with real coefficients (Theorem 2). This new algebraic real system denoted by $\mathcal{T}(S)$ is called the *real transform* of (S). We extend this result to so-called *trapezoidal* fuzzy numbers to get $4s$ equations instead of $3s$ (Section 3.5).

In Section 2, we introduce fuzzy numbers, their different representations, and the transition from one to the other in the case of a fuzzy number with bounded support. In Section 3, we define the real transform of a given fuzzy polynomial equation. We show that, in the triangular fuzzy case, the real transform is a system equivalent to the collected crisp form calculated by previous methods. Finally, the study is extended to the trapezoidal case.

2 FUZZY NUMBERS

After some generalities, this section recalls two classical representations of a fuzzy number that we will use to solve the algebraic fuzzy systems. To go further the reader may be interested in (Dubois et al., 2000). We give some formulas which express the parametric representation of a fuzzy number in function of its tuple representation when the number has a bounded support and bijective spread functions (Proposition 1). These formulas are the key of our algebraic method to solve in $\mathbb{R}^+$ the algebraic fuzzy systems.

2.1 Generalities

Let $\widetilde{n}$ be a fuzzy number and $\mu_{\widetilde{n}}$ its membership function from $\mathbb{R}$ to the real interval $[0,1]$, continuous and satisfying $\mu_{\widetilde{n}}^{-1}(\{1\}) = \{n\}$, where the value n is called the *core* of $\widetilde{n}$. In literature there are more general definitions than the one given above. They include the so-called *trapezoidal* fuzzy numbers, for which the grade of membership equals 1 over an interval of $\mathbb{R}$ containing the core. The chosen definition excludes them for the sake of clarity in our study. Note that most applications make use of non-trapezoidal numbers. However, we will show in Section 3.5 that our analysis, once established, simply extends to the trapezoidal case.

We define the *support* of $\widetilde{n}$ as the support of its membership function, i.e. the set of $a \in \mathbb{R}$ such that $\mu_{\widetilde{n}}(a) \neq 0$. We denote it by $\mathrm{Supp}(\widetilde{n})$.

For a real number r in $[0,1]$, the *r-cut* of $\widetilde{n}$ is the convex set $\widetilde{n}_r = \{x \in \mathbb{R} \mid \mu_{\widetilde{n}}(x) \geq r\}$ when $r \neq 0$ and the 0-cut $\widetilde{n}_0$ is the closure of $\mathrm{Supp}(\widetilde{n})$.

Throughout the paper, we will consider fuzzy numbers with bounded support.

2.2 Tuple Representation

The tuple representation of a fuzzy number with bounded support was proposed in 1978 by D. Dubois and H. Prade in (Dubois and Prade, 1978). In this representation the arithmetic operations have very simple expressions as soon as they are performed within a family described in Definition 1, based on *spread functions*.

A function H defined on the real interval $[0,+\infty[$ with values in the real interval $]-\infty,1]$ is called a *spread function* if $H(0)=1$, $H(1)=0$, H is continue and decreasing on its domain.

Definition 1. *Let L and R be two spread functions. A fuzzy number $\widetilde{n}$ with a bounded support is said of* type *L-R if there exist two positive real numbers α and β such that the membership function $\mu_{\widetilde{n}}$ of $\widetilde{n}$ is given as follows:*

$$\mu_{\widetilde{n}}(x) = \begin{cases} L\left(\frac{n-x}{\alpha}\right) & for\ n-\alpha \leq x < n\ when\ \alpha \neq 0 \\ 1 & for\ x = n \\ R\left(\frac{x-n}{\beta}\right) & for\ n < x \leq n+\beta\ when\ \beta \neq 0 \\ 0 & for\ x \in]-\infty, n-\alpha[\cup]n+\beta,+\infty[. \end{cases}$$

The triplet (n,α,β) is called the tuple representation *of fuzzy number $\widetilde{n}$. Real numbers α and β are respectively called the* left spread *and the* right spread *of $\widetilde{n}$.*

Note that a real number n is identified to the fuzzy number $\widetilde{n}$ with $\alpha=\beta=0$ and $\mathrm{Supp}(\widetilde{n}) = \{n\}$.

We denote by $\mathfrak{F}(L,R)$ the family of fuzzy numbers of type *L-R*. Functions L and R are respectively called the *left spread function* and the *right spread function* of the family $\mathfrak{F}(L,R)$, and by extension the spread functions of $\widetilde{n}$ itself. When $L(x) = R(\frac{x}{k})$, $k>0$, the fuzzy number is said *symmetrical*. Inside a given family $\mathfrak{F}(L,R)$, a tuple (n,α,β) represents an unique element $\widetilde{n}$. The addition is an internal law of $\mathfrak{F}(L,R)$ defined by $(n,\alpha,\beta)+(n',\alpha',\beta') = (n+n',\alpha+\alpha',\beta+\beta')$.

The approximate product "·" is distributive with respect to addition. As we study fuzzy equations with real indeterminates, we consider only products of the form $q \cdot \widetilde{n}$, with $q \in \mathbb{R}$. In this case, the product described by Dubois and Prade becomes exact:

$$q \cdot (n, \alpha, \beta) = \begin{cases} (qn, q\alpha, q\beta) & \text{if } q \geq 0 \\ (qn, -q\beta, -q\alpha) & \text{if } q \leq 0 \end{cases} . \quad (1)$$

Note that the inversion of the spreads that keeps them positive when $q < 0$: $-q\beta$ and $-q\alpha$ are respectively the left spread and the right spread of $q \cdot (n, \alpha, \beta)$. In particular, we have

$$-\widetilde{n} = -1 \cdot (n, \alpha, \beta) = (-n, \beta, \alpha) \quad . \quad (2)$$

2.3 Parametric Representation

The parametric representation introduced in 1986 by R. Goetschel et W. Voxman (Goetschel and Voxman, 1986) allows them to embed all the trapezoidal fuzzy numbers into a topological vector space. The following definition is an adaptation for non-trapezoidal fuzzy numbers:

Definition 2. *The* parametric form *of a fuzzy number $\widetilde{n}$ is an ordered pair $[\underline{n},\ \overline{n}]$ of functions from the real interval $[0,1]$ to $\mathbb{R}$ which satisfy the following conditions:*

(1) $\overline{n}$ is a bounded left continuous non-increasing function on $[0,1]$,

(2) $\underline{n}$ is a bounded left continuous non-decreasing function on $[0,1]$,

(3) $\underline{n}(1) = \overline{n}(1) = n$.

Fuzzy number $\widetilde{n}$ defined by functions $\underline{n}$ and $\overline{n}$ has membership function $\mu_{\widetilde{n}} : \mathbb{R} \to [0,1]$ such that $\mu_{\widetilde{n}}(x) = \sup\{r \mid \underline{n}(r) \leq x \leq \overline{n}(r)\}$.

A fuzzy arithmetic, described in the following lemma, operates on parametric representation. It is coherent with those of the tuple representation given in Section 2.2.

Lemma 1. *(Stefanini and Sorini, 2009) Let $q \in \mathbb{R}$ and $\widetilde{m} = [\underline{m}, \overline{m}]$ and $\widetilde{n} = [\underline{n}, \overline{n}]$ two fuzzy numbers. Then*

1. *$\widetilde{m} = \widetilde{n}$ if and only if $\underline{m}(r) = \underline{n}(r)$ and $\overline{m}(r) = \overline{n}(r)$ for each real $r \in [0,1]$,*
2. *$\widetilde{m} + \widetilde{n} = [\underline{m} + \underline{n},\ \overline{m} + \overline{n}]$,*
3. $q \cdot \widetilde{n} = \begin{cases} [q \cdot \underline{n},\ q \cdot \overline{n}] & \text{if } q \geq 0, \\ [q \cdot \overline{n},\ q \cdot \underline{n}] & \text{if } q \leq 0 \end{cases}$

where, for any function f from $\mathbb{R}$ to $\mathbb{R}$, the product $g = q \cdot f$ represents the function defined as $g(r) = qf(r)$ for each $r \in \mathbb{R}$.

2.4 From Tuple to Parametric Representation

In this paper, we consider polynomials equations with coefficients that are fuzzy numbers of a same family $\mathfrak{F}(L,R)$ satisfying the sufficient requirement that the spread functions L and R are bijective. Our solving method implies to rewrite algebraically each fuzzy coefficient $\widetilde{n} \in \mathfrak{F}(L,R)$ from tuple representation (n, α, β) into parametric representation. The change of representation is given by formulas of Proposition 1 below.

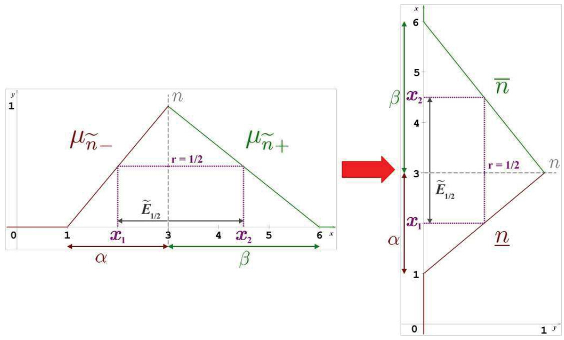

Figure 1: Graph of functions $\underline{3}$ and $\overline{3}$ from the graph of a linear membership function. Here, the *left restriction* (resp. *right restriction*) $\mu_{\widetilde{n}-}$ (resp. $\mu_{\widetilde{n}+}$) is the restriction of $\mu_{\widetilde{n}}$ to the left (resp. right) of the core n.

The parametric representation of $\widetilde{n}$ is strongly related to its r-cuts $\widetilde{n}_r$ since functions $\underline{n}$ and $\overline{n}$ defined by

$$\underline{n}(r) = \inf_{r \in [0,1]} \widetilde{n}_r \quad \text{and} \quad \overline{n}(r) = \sup_{r \in [0,1]} \widetilde{n}_r$$

satisfy the requirements of Definition 2. This relation appears graphically in Figure 1 where $x_1 = \underline{n}(1/2)$ and $x_2 = \overline{n}(1/2)$ for a triangular fuzzy number $\widetilde{3} = (3,2,3)$. The graph of functions $\underline{n}$ and $\overline{n}$ is obtained by a plane rotation of the graph of the membership function followed by a vertical symmetry.

Formally, the transformation is described by the formulas below.

Proposition 1. *Let $\widetilde{n} = (n, \alpha, \beta) \in \mathfrak{F}(L,R)$ where L and R are bijective spread functions. Then the parametric representation $[\underline{n}, \overline{n}]$ of $\widetilde{n}$ satisfies the following formulas:*

$$\begin{cases} \underline{n}(r) = n - \alpha L^{-1}(r) \\ \overline{n}(r) = n + \beta R^{-1}(r) \end{cases} . \quad (3)$$

In particular, when the fuzzy number $\widetilde{n}$ is triangular, we have:

$$\underline{n}(r) = \alpha\, r + n - \alpha \quad \text{and} \quad \overline{n}(r) = -\beta\, r + n + \beta \quad . \quad (4)$$

Proof 1. *For the real number $r \in [0,1]$, Definition 1 implies $r = L\left(\frac{n - \underline{n}(r)}{\alpha}\right) = R\left(\frac{\overline{n}(r) - n}{\beta}\right)$. As L and R are bijective, $\underline{n}(r)$ and $\overline{n}(r)$ satisfy Identity (3) of the proposition.*
In the triangular case, we obtain formula (4) because $L = R = F$ where $F(x) = 1 - x$ is bijective with $F^{-1} = F$.

3 THE REAL TRANSFORM OF A FUZZY EQUATION

The goal of this paper is to find positive solutions of a system of polynomial equations whose coefficients are symmetrical fuzzy numbers belonging to a same family $\mathfrak{F}(L,R)$ where L and R are bijective. It is performed by transforming independently each equation in order to obtain a polynomial system with real coefficients so that it can be solved by algebraic methods. This section is devoted to the transform of only one equation. Note that, in practice, we do not encounter one isolated multivariate equation. For a system reduced to a unique equation, the number of variable is generally reduced to only one too. This particular case can be treated with our method or by others such as (Abbasbandy and Otadi, 2006), and recently (Farahani et al., 2019), but it is not the purpose of this paper.

We will use the following terminology: a variable is said *real* if its represents any real number; a real variable is said *positive* if it represents any positive real number, i.e. belonging to $\mathbb{R}^+$; a k-uplet $(b_1,\ldots,b_k)$ of real variables or real numbers is said *positive* if each component b_i is positive. In this section we consider an algebraic equation (E) with fuzzy coefficients and k real variables $x_1,\ldots,x_k$ also called the indeterminates.

Considering only positive real variables, in Section 3.2 a *crisp form* of (E) is constructed in order to deduce a *collected crisp form* of (E) ; in other words, an algebraic system of equations with real coefficients whose positive solutions are those of (E). In the literature this *collected crisp form* is formed by four equations obtained from (E) by an algorithm that applies only when the fuzzy coefficients are triangular.

Moreover Section 3.3 establishes a formula that provides a particular collected crisp form of (E) formed by only three equations. We call it the *real transform* of (E) and denote it by $\mathcal{T}(E)$. Section 3.4 compares the real transform $\mathcal{T}(E)$ to the usual collected crisp form given in literature for the triangular case. Section 3.5 finally generalizes the results to trapezoidal fuzzy numbers.

3.1 Preliminaries

Let $\boldsymbol{d} = (d_1,\ldots,d_k) \in \mathbb{N}^k$, $\boldsymbol{x} = (x_1,\ldots,x_k)$ and $\boldsymbol{x}^{\boldsymbol{d}} = x_1^{d_1}\cdots x_k^{d_k}$ the *monomial* of multidegree $\boldsymbol{d}$ in the variables $x_1,\ldots,x_k$. In the same way, for $\boldsymbol{a} = (a_1,\ldots,a_k) \in \mathbb{R}^k$, we denote by $\boldsymbol{a}^{\boldsymbol{d}}$ the product $a_1^{d_1}\cdots a_k^{d_k}$. For $\boldsymbol{y} = (y_1,\ldots,y_k)$, we denote by $\boldsymbol{x}\times\boldsymbol{y}$ the classical product $(x_1y_1,\ldots,x_ky_k)$. Note that $(\boldsymbol{x}\times\boldsymbol{y})^{\boldsymbol{d}} = \boldsymbol{x}^{\boldsymbol{d}}\boldsymbol{y}^{\boldsymbol{d}}$.

In this section, we consider the polynomial equation

$$(E): \sum_{\boldsymbol{d}\in\mathrm{Expon}(E)} \widetilde{n_{\boldsymbol{d}}}\, \boldsymbol{x}^{\boldsymbol{d}} = \widetilde{m} \quad , \tag{5}$$

where $\mathrm{Expon}(E)$ is a finite subset of $\mathbb{N}^k$, with $\widetilde{n_{\boldsymbol{d}}} \in \mathfrak{F}(L,R)$ and $\widetilde{m} \in \mathfrak{F}(L,R)\backslash\{(0,0,0)\}$ for all $d \in \mathrm{Expon}(E)$. The fuzzy numbers in (E) are symmetrical and given under their respective tuple representation $\widetilde{m} = (m,\alpha,\beta)$ and $\widetilde{n_{\boldsymbol{d}}} = (n_{\boldsymbol{d}},\alpha_{\boldsymbol{d}},\beta_{\boldsymbol{d}})$. For example, when $k=3$ and (E) is equation $\widetilde{3}x_1^2x_2 + \widetilde{1}x_3^4 = \widetilde{6}$, $\mathrm{Expon}(E)$ is the subset $\{(2,1,0),(0,0,4)\}$ of $\mathbb{N}^3$.

We denote by $\mathrm{Sol}^+(E)$ the set of solutions of (E) in $\mathbb{R}^{+k}$:

$$\mathrm{Sol}^+(E) = \{\, \boldsymbol{a} \in \mathbb{R}^{+k} \mid \sum_{\boldsymbol{d}\in\mathrm{Expon}(E)} \widetilde{n_{\boldsymbol{d}}}\, \boldsymbol{a}^{\boldsymbol{d}} = \widetilde{m} \,\} \quad .$$

We search for $\mathrm{Sol}^+(E)$ by using the r-cuts in order to obtain an algebraic system with real coefficients that can be solved with classical computer algebra methods.

We consider the case where the indeterminates $x_1,\ldots,x_k$ are real and positive. In this context, we seek the formula of the real transform $\mathcal{T}(E)$ of (E). The positive solutions of the real algebraic system $\mathcal{T}(E)$ are exactly the positive solutions of the fuzzy algebraic equation (E).

3.2 Crisp Form of (E) to Find $\mathrm{Sol}^+(E)$

Algebraic solving of fuzzy equation (E) is usually based on the passage of the L-R representation of fuzzy numbers to their parametric representation. In the presentation below we significally strengthen this classical method for triangular fuzzy coefficients by applying it to a generic system and by extending it to more general fuzzy coefficients.

Following Lemma 1, equation (E) rewrites into two equalities on the r-cuts of the left and right members of (E) if all the $\boldsymbol{x}^{\boldsymbol{d}}$, $\boldsymbol{d} \in \mathrm{Expon}(E)$, are supposed to represent reals of the same sign. Indeed, according to Rule (3) of this lemma, the multiplication of a fuzzy number by a scalar q splits into two cases: $q \le 0$ and $q \ge 0$. Thus we search only for the solutions $\boldsymbol{a} \in \mathbb{R}^{+k}$ since the real $q := \boldsymbol{a}^{\boldsymbol{d}}$ is then positive for each $\boldsymbol{d} \in \mathbb{N}^k$.

For $\boldsymbol{a} \in \mathbb{R}^{+k}$, according to Lemma 1, the following equivalence applies:

$$\boldsymbol{a} \in \mathrm{Sol}^+(E) \iff \left[\sum_{\boldsymbol{d}\in\mathrm{Expon}(E)} \underline{n_{\boldsymbol{d}}}(r)\, \boldsymbol{a}^{\boldsymbol{d}},\ \sum_{\boldsymbol{d}\in\mathrm{Expon}(E)} \overline{n_{\boldsymbol{d}}}(r)\, \boldsymbol{a}^{\boldsymbol{d}}\right] = [\underline{m}(r),\overline{m}(r)] \quad . \tag{6}$$

This equivalence leads us to consider $\mathcal{C}(E)$, the following system of two equations with real coefficients and $k+1$ variables $x_1,\ldots,x_k,r$, called the *crisp form* of (E) :

$$\mathcal{C}(E): \left\{ \begin{array}{lcl} \sum\limits_{\boldsymbol{d}\in\mathrm{Expon}(E)} \underline{n_{\boldsymbol{d}}}(r)\,\boldsymbol{x}^{\boldsymbol{d}} & = & \underline{m}(r) \\ \sum\limits_{\boldsymbol{d}\in\mathrm{Expon}(E)} \overline{n_{\boldsymbol{d}}}(r)\,\boldsymbol{x}^{\boldsymbol{d}} & = & \overline{m}(r) \end{array} \right. .$$

Let F be a set of equations in $\mathbb{R}[x_1,\ldots,x_k,r]$. We put

$$\mathrm{Sol}_k^+(F) = \{\boldsymbol{a}\in\mathbb{R}^{+k} \mid \forall r\in[0,1]\ (a_1,\ldots,a_k,r)\in\mathrm{Sol}(F)\}$$

where $\mathrm{Sol}(F)$ is the set of the solutions of F in $\mathbb{R}^{k+1}$. Take $\boldsymbol{a}=(a_1,\ldots,a_k)\in\mathbb{R}^{+k}$. According to Equivalence (6), the k-uplet $\boldsymbol{a}$ is a solution of (E) if and only if for all real $r\in[0,1]$ the $(k+1)$-uplet $(a_1,\ldots,a_k,r)$ is a solution of the crisp form $\mathcal{C}(E)$. In other words, $\mathrm{Sol}_k^+(\mathcal{C}(E))$ is the set of the positive solutions of the fuzzy equation (E):

$$\mathrm{Sol}^+(E) = \mathrm{Sol}_k^+(\mathcal{C}(E)) \quad . \tag{7}$$

In the particular triangular case, the crisp form has two equations with linear expressions w.r.t. the variable r in each side. It is a consequence of formulas (4). The triangular case is easy because the spread functions are equal to $F: x\mapsto 1-x$, with $F^{-1}=F$. The general case, when the spread functions are simply bijective, requires using inversion formulas (3) with two indeterminates instead of only one. This leads to the crisp form with two parameters in the following theorem:

Theorem 1. *Let L and R be two spread functions and*

$$(E): \sum_{\boldsymbol{d}\in Expon(E)} \widetilde{n_{\boldsymbol{d}}}\,\boldsymbol{x}^{\boldsymbol{d}} = \widetilde{m}\ ,$$

be a fuzzy equation with coefficients in the family $\mathfrak{F}(L,R)$ *given by their tuple representations as follows:* $\widetilde{m}=(m,\alpha,\beta)$ *and* $\widetilde{n_{\boldsymbol{d}}}=(n_{\boldsymbol{d}},\alpha_{\boldsymbol{d}},\beta_{\boldsymbol{d}})$ *for* $\boldsymbol{d}\in$ *Expon*(E). *If the spread functions L and R are bijective then the crisp form of* (E) *is given by:*

$$\mathcal{C}(E): \left\{ \begin{array}{lcl} \sum\limits_{\boldsymbol{d}} n_{\boldsymbol{d}}\,\boldsymbol{x}^{\boldsymbol{d}} - m + (\alpha - \sum\limits_{\boldsymbol{d}} \alpha_{\boldsymbol{d}}\,\boldsymbol{x}^{\boldsymbol{d}})\,u & = & 0 \\ \sum\limits_{\boldsymbol{d}} n_{\boldsymbol{d}}\,\boldsymbol{x}^{\boldsymbol{d}} - m + (-\beta + \sum\limits_{\boldsymbol{d}} \beta_{\boldsymbol{d}}\,\boldsymbol{x}^{\boldsymbol{d}})\,v & = & 0\ , \end{array} \right. \tag{8}$$

where $u=L^{-1}(r)$ *and* $v=R^{-1}(r)$ *for all* $r\in[0,1]$. *For* $\boldsymbol{a}\in\mathbb{R}^{+k}$, *we have* $\boldsymbol{a}\in\mathrm{Sol}^+(E)$ *if and only if, for all* $r\in[0,1]$, *system* (8) *is satisfied by the* $(k+2)$*-uplet* $(a_1,\ldots,a_k,L^{-1}(r),R^{-1}(r))$.

Proof 2. *By definition, a spread function H sends* $[0,1]$ *to itself and if moreover H is bijective then its inverse* H^{-1} *is continue and decreasing with* $H^{-1}(1)=0$ *and* $H^{-1}(0)=1$. *Suppose that the spread functions L and R are bijective. As each r belongs to* $[0,1]$, *we can put* $u=L^{-1}(r)$ *and* $v=R^{-1}(r)$. *When r runs throughout* $[0,1]$ *in the increasing sens, the parameters u and v run throughout the same interval* $[0,1]$ *in the decreasing sens. According to formulas (3), the parametric form of the coefficients of the equation are given by*

$$\begin{array}{lclclcl} \underline{n_{\boldsymbol{d}}}(r) & = & n_{\boldsymbol{d}} - \alpha_{\boldsymbol{d}}\,u & , & \overline{n_{\boldsymbol{d}}}(r) & = & n_{\boldsymbol{d}} + \beta_{\boldsymbol{d}}\,v \\ \underline{m}(r) & = & m - \alpha u & , & \overline{m}(r) & = & m + \beta v \end{array} . \tag{9}$$

for $\boldsymbol{d}\in$ *Expon*(E). *Then the crisp form* $\mathcal{C}(E)$ *of* (E) *given in* (7) *is written as a system of two equations with* $k+2$ *variables* $x_1,\ldots,x_k,u,v$, *where u and v are dependent on each other:*

$$\mathcal{C}(E): \left\{ \begin{array}{l} \sum\limits_{\boldsymbol{d}} n_{\boldsymbol{d}}\,\boldsymbol{x}^{\boldsymbol{d}} - \alpha_{\boldsymbol{d}}\,u\,\boldsymbol{x}^{\boldsymbol{d}} = m - \alpha u \\ \sum\limits_{\boldsymbol{d}} n_{\boldsymbol{d}}\,\boldsymbol{x}^{\boldsymbol{d}} + \beta_{\boldsymbol{d}}\,v\,\boldsymbol{x}^{\boldsymbol{d}} = m + \beta v \end{array} \right.$$

By collecting all the terms on the left-hand-side of the two equations, we find the crisp form expressed as in the form (8) *of the theorem. Last assertion in the theorem about* $\mathrm{Sol}^+(E)$ *follows directly from equality* (7).

Note that Theorem 1 only requires that the restrictions on $[0,1]$ of the two spread functions L and R are bijective.

Our approach allows at the same time to improve and to generalize the methods known so far. For instance, results in (Molai et al., 2013) and (Boroujeni et al., 2016) are restricted to triangular fuzzy numbers. Indeed, the crisp form of (E) with two parameters $u=L^{-1}(r)$ and $v=R^{-1}(r)$ given in Identity (8) is a generalization of the crisp form known in triangular case with only one parameter r where $r\in[0,1]$.

In the aforementioned articles, for each problem to be solved, the algorithm computes the system $\mathcal{C}(E)$ in variables $x_1,\ldots,x_k,r$, which is linear w.r.t. r. Then it is rewritten into an equivalent system of four algebraic equations in $x_1,\ldots,x_k$ with real coefficients called *collected crisp form* of (E). In next section, we will show how to get a particular collected crisp form reduced to three equations, for symmetrical fuzzy coefficients of any family $\mathfrak{F}(L,R)$ such that the spread functions L and R are bijective. This is the real transform of (E). In addition, we explicitly give its formulation from (E).

3.3 The Real Transform and the Positive Real Solutions of (E)

We define here the real transform of a fuzzy equation (E) and show that its positive real solutions are also those of (E).

Definition 3. *Let L and R be two spread functions and*

$$(E): \sum_{\boldsymbol{d}\in Expon(E)} \widetilde{n_{\boldsymbol{d}}}\,\boldsymbol{x}^{\boldsymbol{d}} = \widetilde{m}\ ,$$

a fuzzy equation with symmetrical coefficients in the family $\mathfrak{F}(L,R)$ given by their representations in tuple as follows: $\widetilde{n}_{\boldsymbol{d}} = (n_{\boldsymbol{d}}, \alpha_{\boldsymbol{d}}, \beta_{\boldsymbol{d}})$ ($\boldsymbol{d} \in Expon(E)$) and $\widetilde{m} = (m, \alpha, \beta)$. The real transform *$\mathcal{T}(E)$ of (E) is the following polynomial system over $\mathbb{R}$:*

$$\mathcal{T}(E) : \begin{cases} \sum\limits_{\boldsymbol{d} \in Expon(E)} n_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} & = & m \\ \sum\limits_{\boldsymbol{d} \in Expon(E)} \alpha_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} & = & \alpha \\ \sum\limits_{\boldsymbol{d} \in Expon(E)} \beta_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} & = & \beta\,. \end{cases} \tag{10}$$

This definition naturally extends to a system (S) of fuzzy equations such as (E). We denote by $\mathcal{T}(S)$ its real transform, i.e. the system formed by the real transforms of the equations in (S).

Theorem 2. *According to Definition 3, if the two spread functions L and R are bijective then the set of positive real solutions of (E) equals the one of its real transform; in other words:*

$$\mathrm{Sol}^+(E) = \mathrm{Sol}^+(\mathcal{T}(E)) \quad .$$

Proof 3. *Let be $\boldsymbol{a} \in \mathbb{R}^{+k}$. As the spread functions L and R are bijective, we can apply Theorem 1. According to this theorem, we know that $\boldsymbol{a} \in \mathrm{Sol}^+(E)$ if and only if, for all $r \in [0,1]$, the crisp form of (E) expressed in (8) is satisfied by the $(k+2)$-uplet $(a_1, \ldots, a_k, L^{-1}(r), R^{-1}(r))$.*

With $r = 1$, we have $u = L^{-1}(1) = 0$. Then $\boldsymbol{a} \in \mathrm{Sol}^+(E)$ satisfies the equation $\sum_d n_{\boldsymbol{d}} \boldsymbol{x}^{\boldsymbol{d}} = m$. Note that when $r = 1$ we have $v = 0$ too because $R(0) = 1$, and we find the same equation and not a second one. This is why we obtain three equations instead of four. Then, by taking $r = 0$ we have $u = L^{-1}(0) = 1$ and $v = R^{-1}(0) = 1$. By replacing in (8) the expression $\sum_d n_{\boldsymbol{d}} \boldsymbol{x}^{\boldsymbol{d}} - m$ by 0 and each variable u and v by 1, we deduce that a positive solution of (E) is also a positive solution of the real transform $\mathcal{T}(E)$ of (E).

For the inverse inclusion, consider the crisp form $\mathcal{C}(E)$ as a polynomial system in the variables of $\boldsymbol{x}$ and with coefficients in the ring $\mathbb{R}[u,v]$. Any solution $(a_1, \ldots, a_k) \in \mathbb{R}^k$ of $\mathcal{T}(E)$ is also a solution of $\mathcal{C}(E)$ in $\mathbb{R}^k$ whatever the parameters u and v may be in the interval $[0,1]$. Obviously it remains true when they are furthermore connected by the constraint $L^{-1}(u) = R^{-1}(v) \in [0,1]$. Hence any positive real solution of the real transform $\mathcal{T}(E)$ is also a positive real solution of the fuzzy equation (E).

Theorem 2 ensures that finding the positive real roots of (E) amounts to finding the positive real roots of its real transform $\mathcal{T}(E)$. Therefore it is no use to develop intermediate computations on parametric representation like the previous methods did in the specific triangular case.

3.4 Comparison with Previous Methods in the Triangular Case

Consider a system (S) formed by s polynomial equations with symmetrical fuzzy coefficients. In the specific case of triangular fuzzy numbers as coefficients, the authors of (Molai et al., 2013) and (Boroujeni et al., 2016) compute a collected crisp form of (S) formed by $4s$ real algebraic equations. In this part we are interested in the relationship between their collected crisp form with $4s$ equations and our collected crisp system with $3s$ equations, that is the real transform of (S). For both systems, the positive real solutions of each of these systems are also those of (S). It is the principle of any collected crisp form of (S).

Consider below the system F_1 of Section 6 in (Boroujeni et al., 2016):

$$F_1 : \begin{cases} (2,1,1)xy + (3,1,1)x^2y^2 + (2,1,1)x^3y^3 = (7,3,3) \\ (5,1,1)xy + (2,3,1)x^2y^2 + (2,2,1)x^3y^3 = (9,6,3). \end{cases}$$

Applied to first equation, the algorithm proposed in (Boroujeni et al., 2016) produces the following collected crisp form:

$$\begin{cases} xy + x^3y^3 - 3 + x^2y^2 = 0, \\ xy + 2x^2y^2 - 4 + x^3y^3 = 0 \\ -xy - x^3y^3 + 3 - x^2y^2 = 0 \\ 3xy + 4x^2y^2 - 10 + 3x^3y^3 = 0 \quad ; \end{cases}$$

and it produces the following collected crisp form of the second equation of F_1:

$$\begin{cases} xy + 3x^2y^2 + 2x^3y^3 - 6 = 0, \\ 4xy - x^2y^2 - 3 = 0, \\ -xy - x^3y^3 + 3 - x^2y^2 = 0, \\ 6xy + 3x^2y^2 - 12 + 3x^3y^3 = 0 \quad . \end{cases}$$

Call T_1 the system formed of the eight preceding equations.

Furthermore, by applying to F_1 our formula (10) defining the real transform, we get $\mathcal{T}(F_1)$, the following system of six equations:

$$\mathcal{T}(F_1) : \begin{cases} 2xy + 3x^2y^2 + 2x^3y^3 = 7 \\ xy + x^2y^2 + x^3y^3 = 3 \\ xy + x^2y^2 + x^3y^3 = 3 \\ 5xy + 2x^2y^2 + 2x^3y^3 = 9 \\ xy + 3x^2y^2 + 2x^3y^3 = 6 \\ xy + x^2y^2 + x^3y^3 = 3 \quad . \end{cases}$$

An easy computation shows the equivalence of the systems T_1 and $\mathcal{T}(F_1)$, whose set of solutions is $\{(x,y) \in \mathbb{R} \mid xy = 1\}$. This phenomenon of equivalence between both approaches may be explained in a very general way as we show below by considering the classical computation of the collected crisp form obtained by an application of the algorithm of (Boroujeni et al., 2016) on the generic equation (5) of (E).

Let $\widetilde{m} = (n,\alpha,\beta)$ and $\widetilde{n_{\boldsymbol{d}}} = (n_{\boldsymbol{d}},\alpha_{\boldsymbol{d}},\beta_{\boldsymbol{d}})$ ($\boldsymbol{d} \in \text{Expon}(E)$) be the respective tuple representations of the fuzzy coefficients of (E) that are assumed to be triangular. According to formulas (4), in the triangular case the r-cuts are given by

$$\begin{array}{rcl} \widetilde{n_{\boldsymbol{d}}}(r) & = & [\alpha_{\boldsymbol{d}} r + n_{\boldsymbol{d}} - \alpha_{\boldsymbol{d}}\,, -\beta_{\boldsymbol{d}} r + n_{\boldsymbol{d}} + \beta_{\boldsymbol{d}}] \\ \widetilde{m}(r) & = & [\alpha r + m - \alpha\,, -\beta r + m + \beta]\ . \end{array}$$

for $\boldsymbol{d} \in \text{Expon}(E)$. For a triangular fuzzy number, $L = R = F$, where $F(x) = F^{-1}(x) = 1 - x$, being known, the previous methods replace directly $L^{-1}(r)$ and $R^{-1}(r)$ by their expression in the variable r in the equations. That's how they end up in the crisp form of (E) below expressed as two polynomials in the variable r:

$$\mathcal{C}(E) : \begin{cases} (\sum_{\boldsymbol{d}} \alpha_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} - \alpha) r + \sum_{\boldsymbol{d}} (n_{\boldsymbol{d}} - \alpha_{\boldsymbol{d}})\, \boldsymbol{x}^{\boldsymbol{d}} - m + \alpha = 0 \\ (\beta - \sum_{\boldsymbol{d}} \beta_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}}) r + \sum_{\boldsymbol{d}} (n_{\boldsymbol{d}} + \beta_{\boldsymbol{d}})\, \boldsymbol{x}^{\boldsymbol{d}} - m - \beta = 0. \end{cases}$$

A k-uplet $(x_1,\ldots,x_k) \in \mathbb{R}^k$ is a solution of $\mathcal{C}(E)$ for all $r \in [0,1]$ if and only if each coefficient w.r.t. the variable r of these independent equations is zero. The collected crisp form of (E) is therefore written

$$\begin{cases} \sum_{\boldsymbol{d}} \alpha_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} & = & \alpha \\ \sum_{\boldsymbol{d}} (n_{\boldsymbol{d}} - \alpha_{\boldsymbol{d}})\, \boldsymbol{x}^{\boldsymbol{d}} & = & m - \alpha \\ \sum_{\boldsymbol{d}} \beta_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} & = & \beta \\ \sum_{\boldsymbol{d}} (n_{\boldsymbol{d}} + \beta_{\boldsymbol{d}})\, \boldsymbol{x}^{\boldsymbol{d}} & = & m + \beta\ . \end{cases}$$

By applying this transformation to each equation in the system F_1, we find the collected crisp form T_1 of our example. For the generic equation (E), by injecting the first equation into the second one and by noting that the last equation is the sum of the three other ones, we obtain the real transform $\mathcal{T}(E)$ with three equations defined in (10).

3.5 Case of Trapezoidal Fuzzy Numbers

In this section, we extend the definition of fuzzy numbers to *trapezoidal* fuzzy numbers by allowing $\mu_{\widetilde{n}}^{-1}(\{1\})$ to an interval $[a,b]$. As mentioned in Section 2.1 our results adapt to symmetrical trapezoidal fuzzy numbers with bounded support.

In this context, a fuzzy number $\widetilde{n}$ with bounded support is of type *L-R* if its membership function $\mu_{\widetilde{n}}$ has the following form:

$$\mu_{\widetilde{n}}(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right) & \text{for } a - \alpha \le x < a \text{ when } \alpha \neq 0 \\ 1 & \text{for } x \in [a,b] \\ R\left(\frac{x-n}{\beta}\right) & \text{for } b < x \le b + \beta \text{ when } \beta \neq 0 \\ 0 & \text{for } x \in]-\infty, a-\alpha[\cup]b+\beta, +\infty[. \end{cases}$$

Then the tuple representation of the fuzzy number $\widetilde{n}$ is the quadruplet (a,b,α,β). The expression of the parametric representation given in Proposition 1 takes the following form for a trapezoidal number of type $L-R$ whose spread functions L and R are bijective:

$$\begin{cases} \underline{n}(r) = a - \alpha L^{-1}(r) \\ \overline{n}(r) = b + \beta R^{-1}(r) \quad . \end{cases} \tag{11}$$

When the equation $(E) : \sum_{\boldsymbol{d} \in \text{Expon}(E)} \widetilde{n_{\boldsymbol{d}}}\, \boldsymbol{x}^{\boldsymbol{d}} = \widetilde{m}$ has symmetrical trapezoidal fuzzy coefficients of type *L-R*, where $\widetilde{n_{\boldsymbol{d}}} = (a_d, b_d, \alpha_d, \beta_d)$ and $\widetilde{m} = (a,b,\alpha,\beta)$ are the tuple representations of the coefficients, the parametric forms (9) given in the proof of Theorem 1 become

$$\begin{array}{rclcrcl} \underline{n_{\boldsymbol{d}}}(r) & = & a_{\boldsymbol{d}} - \alpha_{\boldsymbol{d}}\, u & , & \overline{n_{\boldsymbol{d}}}(r) & = & b_{\boldsymbol{d}} + \beta_{\boldsymbol{d}}\, v \\ \underline{m}(r) & = & a - \alpha u & , & \overline{m}(r) & = & b + \beta v\ . \end{array}$$

for $\boldsymbol{d} \in \text{Expon}(E)$. Applying the argument of Section 3.3 (here $L(1) = L(1) = 0$ and $R(0) = R(0) = 1$), we obtain in the same way a real transform of (E), but this time with four equations:

$$\mathcal{T}(E) : \begin{cases} \sum_{\boldsymbol{d}} a_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} & = & a \\ \sum_{\boldsymbol{d}} b_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} & = & b \\ \sum_{\boldsymbol{d}} \alpha_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} & = & \alpha \\ \sum_{\boldsymbol{d}} \beta_{\boldsymbol{d}}\, \boldsymbol{x}^{\boldsymbol{d}} & = & \beta\ . \end{cases} \tag{12}$$

Consequently the use of the real transform for solving polynomial fuzzy systems will directly transpose to systems with symmetrical trapezoidal fuzzy coefficients.

The real transform of a fuzzy system (S) of s equations, denoted $\mathcal{T}(S)$, with $4s$ instead of $3s$ equations, is obtained by slightly applying formula (12) to each equation of (S).

4 CONCLUSION

Up to now, given a fuzzy system (S) of s equations and k indeterminates, the existing algebraic methods have performed computations with the parametric representation of the coefficients to obtain the collected crisp form of (S) formed by $4s$ real equations. We show that these computations are superfluous and exhibit a formula that defines an equivalent system with $3s$ real equations. We call it the real transform $\mathcal{T}(S)$ of the system (S). As a main property, it has the same positive solutions as (S) (Theorem 2).

Unlike the previous methods that were restricted to triangular fuzzy numbers, our results apply to symmetrical fuzzy numbers of any family $\mathfrak{F}(L,R)$ where the spread functions L and R are bijective. Moreover there is no use to compute the inverse of the spread functions since the real transform is a universal formula independent from L and R.

Further work will explore how to obtain the whole set of real zeros of (E), not only positive zeros. The problem when computing with real variables and fuzzy numbers is intrinsic to fuzzy numbers since the product by a real scalar is expressed differently depending on the sign of this scalar.

One idea to work around the problem of unknown sign of $\boldsymbol{x^d}$ is to only focus on positive solutions by putting back the issue on the fuzzy coefficients. We solve the system by introducing an artificial k-uplet of signs $I \in \{-1,1\}^k$, and we replace $\widetilde{n_d}\boldsymbol{x^d}$ where $\boldsymbol{x}$ represents any real by $\boldsymbol{I^d}\widetilde{n_d}|\boldsymbol{x}|^{\boldsymbol{d}}$ which equals $\widetilde{n_d}|\boldsymbol{x}|^{\boldsymbol{d}}$ or $-\widetilde{n_d}|\boldsymbol{x}|^{\boldsymbol{d}}$ depending on the sign of $\boldsymbol{x^d}$, where $|\boldsymbol{x}|$ is positive. The 2^k possible k-uplets for I induce the same number of induced equations $E(I)$. Hence, we recover the solutions of (E) from the positive solutions of its 2^k induced equations $E(I)$.

To obtain the real solutions of (E), it will be necessary and sufficient to collect the positive real solutions of the 2^k real transforms $\mathcal{T}(E(I))$. In practice, since the equations $E(I)$ are not pairwise distinct, a strategy will be needed to reduce the number of induced systems $\mathcal{T}(E(I))$ to solve, in order to implement an optimized algorithm that automatizes the research of solutions by avoiding the studies of signs needed in previous methods.

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